LETTERS TO THE EDITOR

Existence of extraordinary zero-curvature slowness curve in anisotropic elastic media (L)

Litian Wang and Kent G. Ryne
Østfold University College, Engineering, 1757 Halden, Norway

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Acoustic wave propagation in elastic media is characterized by the slowness surface. The slowness surface consists of three sheets associated with three modes of wave propagation and the two outer sheets can have zero-curvature locally. It is shown that the outmost sheet can admit extraordinary zero-curvature and the slowness curve can appear as a straight line locally. Using the perturbation method, the conditions for the extraordinary zero-curvature are derived analytically without violating the thermodynamic condition for elastic media. The results can be applied to crystals with higher symmetry and to the study of phonon focusing and surface waves.
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The slowness surface that characterizes acoustic wave propagation in elastic media plays a central role in understanding the anisotropic phenomena associated with bulk waves, phonon focusing, and surface waves.[1–4] As an important parameter, curvature of the slowness surface has been studied extensively in describing caustics in the phonon imaging where the caustics are related to parabolic lines on the slowness surface.[4] Since the slowness surface consists of three sheets associated with three-mode wave propagation and the two outer sheets can have zero-curvature, an interesting question can be raised as to whether the outermost sheet can have extraordinary zero-curvature. The extraordinary zero-curvature refers to the case where the slowness curve becomes straight locally in certain directions. The present work is to prove the existence of extraordinary zero-curvature in the vicinity of a twofold symmetry axis in monoclinic elastic media.

Wave propagation in anisotropic elastic media is governed by the Christoffel equation.[1–3] For a given propagation direction $\mathbf{k}$, the Christoffel equation is given by

$$\Gamma \mathbf{A} = \gamma \mathbf{A}, \tag{1}$$

where $\Gamma = k_i c_{ijkl} k_l$, $\gamma = \rho v^2$, and c_{ijkl} denotes the elastic stiffness tensor. Being a three-dimensional matrix, Γ yields three eigenvalues in terms of $\rho v^2(\mathbf{k})$, leaving three slowness surfaces $s(\mathbf{k}) = 1/v(\mathbf{k})$ associated with $|\mathbf{k}| = 1$. The innermost sheet describes the quasi-longitudinally polarized waves and is convex globally, while the two outersheets represent the quasi-transversely polarized waves and can be concave locally. The local concavity plays a central role in the study of phonon focusing and surface waves in anisotropic elastic media.[4–8]

Without losing generality, we consider a monoclinic medium with a twofold symmetry along the $\mathbf{e}_z$ axis. Monoclinic media are usually described by 13 elastic stiffness constants, and only 12 of them are independent. By properly choosing the coordinate system, we can transform the elastic tensor into the one with 12 elastic constants leaving $c_{45} = 0$. This is done by setting $\mathbf{e}_x$ parallel to the polarization of the transverse mode defined by $\rho v^2 = c_{55}$ with $\mathbf{k} = \mathbf{e}_z$.[3] The following discussion is confined to the wave propagation along the $\mathbf{e}_z$ axis (the twofold symmetry axis), and the normal curvature with respect to $\mathbf{e}_z - \mathbf{e}_x$ plane. Defining the wave vector with $\mathbf{k} = \cos\theta \mathbf{e}_z + \sin\theta \mathbf{e}_x$, the matrix Γ can then be formulated by

$$\begin{pmatrix} c_{55} + \Delta_{15}\sin^2\theta & c_{16}\sin^2\theta & d_{13}\cos\theta\sin\theta \\ c_{16}\sin^2\theta & c_{44} + \Delta_{64}\sin^2\theta & c_{36}\cos\theta\sin\theta \\ d_{13}\cos\theta\sin\theta & c_{36}\cos\theta\sin\theta & c_{33} + \Delta_{53}\sin^2\theta \end{pmatrix}, \tag{2}$$

where $\Delta_{IJ} = c_{II} - c_{JJ}$ and $d_{13} = c_{13} + c_{55}$. By normalizing elastic constants against $c_{11}/\sqrt{\rho}$, the three eigenvalues γ_α give three slowness curves with $s_\alpha(\theta) = v_\alpha^{-1}(\theta) = \gamma_\alpha^{-1/2}(\theta)$.

It is well known that along the $\mathbf{e}_z$ axis the two outer slowness curves can have zero-curvature. However, whether the outermost slowness curve can have extraordinary zero-curvature has not been studied because its significance was not fully recognized. Geometrically, the extraordinary zero-curvature, if it is possible, would require vanishing first and/or second derivatives of the curvature. Because of the twofold symmetry about the $\mathbf{e}_z$ axis, the slowness curves are symmetric with respect to the $\mathbf{e}_z$ axis; it leaves $d\gamma/d\theta$ and $d^3\gamma/d\theta^3$ zero simultaneously. Since the slowness curves can be expressed in terms of one parameter θ, the normal curvature for the slowness curve and its second derivative in the vicinity of $\mathbf{e}_z$ axis can be defined by

$$k = -\gamma^{-1/2}\left(\gamma + \tfrac{1}{2}\gamma''\right)\Big|_{\theta=0}, \tag{3}$$

$$k'' = 4\gamma^{1/2}\left(\gamma - \tfrac{1}{8}\gamma^{(4)}\right)\Big|_{\theta=0}. \tag{4}$$

The main difficulty is to derive general analytical expressions for the slowness curves. In the vicinity of the $\mathbf{e}_z$ axis, by rewriting the matrix Γ as $\Gamma^0 + V(\theta)$, where $\Gamma^0 = \mathrm{diag}\,[c_{55}, c_{44}, c_{33}]$, we can obtain explicit expression for the slowness by using the perturbation method. We expanded the matrix $V(\theta)$ in the vicinity of the $\mathbf{e}_z$ axis ($\theta = 0$) and applied